BULLETIN OF THE BRAZILIAN MATHEMATICAL SOCIETY

Bull Braz Math Soc, New Series 34(3), 437-455

Restoration of Gibbsianness for projected and FKG renormalized measures

Roberto Fernández, Arnaud Le Ny and Frank Redig

Abstract. We restore part of the thermodynamic formalism for some renormalized measures that are known to be non-Gibbsian. We determine a necessary and sufficient condition for consistency with a specification that is quasilocal only in a fixed direction. This condition is then applied to models with FKG monotonicity and to models with appropriate "directional continuity rates", in particular to (noisy) decimations or projections of the Ising model. In this way we establish: (i) the validity of the "second part" of the variational principle for projected and FKG block-renormalized measures, and (ii) the almost quasilocality of FKG block-renormalized "+" and "−" measures.

Keywords: Gibbs vs non-Gibbs, generalized Gibbs measures, almost quasilocality, thermodynamic formalism.

Mathematical subject classification: Primary: 60G60; Secondary: 82B20, 82B30.

1 Introduction

The problem of *restoration of Gibbsianness* refers to the extension of Gibbsian theory to non-Gibbsian measures observed in statistical mechanics. The latter include measures obtained through renormalization transformations (see [8] and references therein), joint measures of disordered spin systems [19, 20], and measures obtained as a result of stochastic evolutions of Gibbs measures [7]. See [5, 6, 9, 10] for reviews. The goal of the restoration program is the determination of appropriate, more general, classes of measures which would satisfy, in particular, a Gibbsian-like thermodynamic formalism based on the variational principle. Two classes of measures have been introduced so far

(1) *Weak Gibbsian measures* [4, 2, 28]. These are measures that admit an almost sure Boltzman-Gibbs description, that is, whose finite-volume con-

Received 7 March 2003.

ditional probabilities can be written in terms of an interaction potential which is summable on a set of full measure.

(2) *Almost quasilocal measures* [12, 23]. These are measures whose finite-volume conditional probability are continuous functions of the exterior configuration, except possibly at a set of measure zero.

Almost quasilocality is a property strictly stronger than weak Gibbsianness [27, 20, 26]. Most of the measures obtained by block renormalization transformations and by projections have been shown to be weak Gibbsian. The issue of their almost quasilocality has been addressed only very recently [11]. In the context of weak Gibbsianness, the thermodynamic approach has been discussed in [25, 27], where the following is established for projections and block-renormalized measures: (i) existence of thermodynamic functionals, and (ii) validity of a relation between "consistency with the same specification" and "zero relative entropy", for classes of measures with suitable support properties. The support restriction is essential. Indeed, there are examples of weak Gibbsian measures with no common version of conditional probabilities and zero relative entropy [21].

In this paper we study the variational approach for almost quasilocal measures. We first spell out how the work of Pfister [30] on asymptotically decoupled measures, settles down the issue of existence of conjugated thermodynamic potentials for block-renormalized measures. This fact, called below *the specification-independent variational principle*, holds quite generally, without any additional hypothesis and independently of any Gibbs-restoration approach. We then establish conditions for the validity of the usual (specification-dependent) variational principle in statistical mechanics. More specifically, we study the so-called "second part" of this principle, namely when a zero relative entropy density implies consistency with the same specification. Our main theorem (Theorem 3.3) relates this implication to concentration properties of finite-volume relative densities. We discuss two types of applications. First we consider specifications that are monotonicity preserving in FKG sense [13]. In Corollary 3.5 we show implications of the previous theorem regarding the almost quasilocality of the consistent measures. In particular, block-transformed "$+$" and "$-$" measures are quasilocal in this FKG setting (Corollary 3.6). For noisy decimations, this strengthens previous weak Gibbsianness results [2, 29]. In our second application (Proposition 3.12) the monotonicity hypothesis is replaced by the existence of appropriate "continuity rates". From this proposition we obtain, in particular, that for projections to a line "zero entropy density $\Longrightarrow$ consistency with the same specification" without any support assumption. The present results have been already exploited in [11] in the particular case of decimations and projections to a layer. We take the opportunity offered by the school to present the general framework in which our results apply. In comparison with the laborious weak-Gibbisianness estimations based on cluster expansions, our proofs are very short. Yet, some of the results are stronger. This fact reinforces our belief that almost quasilocality is a notion naturally related to the variational approach. As these results illustrate, important aspects of the variational principle can be more simply related to properties of specifications, without having to rely on rather detailed descriptions of weak Gibbsian potentials. The recent work in [21] brings further support to this point of view.

2 Basic definitions and notation

We start by summarizing some basic notions for the sake of completeness. As general reference we mention [14]. See also [8], Section 2, for a streamlined exposition.

2.1 Quasilocality, specifications, consistent measures

We consider configuration spaces $\Omega = \Omega_0^{\mathbb{L}}$ with Ω_0 finite and $\mathbb{L}$ countable (typically $\mathbb{L} = \mathbb{Z}^d$), equipped with the product discrete topology and the product Borel σ-algebra $\mathcal{F}$. More generally, for (finite or infinite) subsets Λ of $\mathbb{L}$ we consider the corresponding measurable spaces $(\Omega_\Lambda, \mathcal{F}_\Lambda)$, where $\Omega_\Lambda = \{-1, 1\}^\Lambda$. For any $\omega \in \Omega$, ω_Λ denotes its projection on Ω_Λ. We denote by $\mathcal{S}$ the set of finite subsets of $\mathbb{L}$. A function $f : \Omega \to \mathbb{R}$ is **local** if there exists a finite set Δ such that $\omega_\Delta = \sigma_\Delta$ implies $f(\omega) = f(\sigma)$. The set of local functions is denoted $\mathcal{F}_{loc}$.

Definition 2.1. *Let* $f : \Omega \to \mathbb{R}$.

(i) f *is* **quasilocal** *if it is the uniform limit of local functions, that is, if*

$$\lim_{\Lambda \uparrow \mathbb{L}} \sup_{\substack{\sigma,\omega: \\ \sigma_\Lambda = \omega_\Lambda}} \Big| f(\sigma) - f(\omega) \Big| \;=\; 0 \,. \tag{2.2}$$

[The notation $\Lambda \uparrow \mathbb{L}$ *means convergence along a net directed by inclusion.]*

(ii) f *is* **quasilocal in the direction** $\theta \in \Omega$ *if*

$$\lim_{\Lambda \uparrow \mathbb{L}} \Big| f(\omega_\Lambda \theta_{\Lambda^c}) - f(\omega) \Big| \;=\; 0 \tag{2.3}$$

for each $\omega \in \Omega$. *[No uniformity required.]*

Remark 2.4. In the present setting (product of finite single-spin spaces) quasi-locality is equivalent to continuity and uniform continuity. This follows from Stone-Weierstrass plus the fact that local functions are continuous for discrete spaces.

Remark 2.5. The pointwise analogues of (2.2) and (2.3) are the following:
(i) f is *quasilocal at* ω if $\lim_{\Lambda\uparrow\mathbb{L}} \sup_{\sigma,\eta}\Big|f(\omega_\Lambda\sigma_{\Lambda^c}) - f(\omega_\Lambda\eta_{\Lambda^c})\Big| = 0$;
(ii) f is *quasilocal at* ω *in the direction* θ if (2.3) holds only for this ω.
We shall not resort to these notions, but let us point out that the previous remark is no longer valid at the pointwise level: A function can be quasilocal in every direction at a certain ω (that is, continuous at ω) and fail to be quasilocal at ω.

Here is an example illustrating the last remark. Let $d = 1$ and, for a fixed $\omega \in \Omega$, choose a countable family $\chi^{(m)}$ of configurations such that $\chi^{(m)}_{[-n,n]^c} \neq \chi^{(m')}_{[-n,n]^c}$ for all $n \in \mathbb{N}$, if $m \neq m'$, and such that $\chi^{(m)}_0 \neq \omega_0$ for all m. Define

$$f(\eta) \;=\; \begin{cases} m/(n+m) \text{ if } \eta = \omega_{[-n,n]}\,\chi^{(m)}_{[-n,n]^c} \text{ for some } m,n \in \mathbb{N} \\ 0 \qquad\qquad\quad \text{otherwise}\,. \end{cases} \tag{2.6}$$

We see that, for all $\sigma \in \Omega$,

$$\lim_{\Lambda\uparrow\mathbb{L}} f(\omega_\Lambda\sigma_{\Lambda^c}) \to 0 = f(\omega). \tag{2.7}$$

Hence f is quasilocal at ω in every direction. However,

$$\sup_{\sigma,\eta}\Big|f(\omega_{[-n,n]}\sigma_{[-n,n]^c}) - f(\omega_{[-n,n]}\eta_{[-n,n]^c})\Big| \;=\; \sup_m \frac{m}{n+m} \;=\; 1\,. \tag{2.8}$$

So f is not quasilocal at ω.

Definition 2.9. *A* **specification** *on* $(\Omega, \mathcal{F})$ *is a family* $\gamma = \{\gamma_\Lambda, \Lambda \in S\}$ *of probability kernels on* $(\Omega, \mathcal{F})$ *that are*

(I) Proper*:* $\forall B \in \mathcal{F}_{\Lambda^c}$, $\gamma_\Lambda(B|\omega) = \mathbf{1}_B(\omega)$.

(II) Consistent*: If* $\Lambda \subset \Lambda'$ *are finite sets, then* $\gamma_{\Lambda'}\gamma_\Lambda = \gamma_{\Lambda'}$.

[We adopt the "conditional-probability" notation, that is, $\gamma_\Lambda(A|\cdot)$ is $\mathcal{F}_{\Lambda^c}$-measurable $\forall A \in \mathcal{F}$, and $\gamma_\Lambda(\cdot|\omega)$ is a probability measure on $(\Omega, \mathcal{F})$ $\forall\omega \in \Omega$.] The notation $\gamma_{\Lambda'}\gamma_\Lambda$ refers to the natural composition of probability kernels:

$(\gamma_{\Lambda'}\gamma_\Lambda)(A|\omega) = \int_\Omega \gamma_\Lambda(A|\omega')\gamma_{\Lambda'}(d\omega'|\omega)$. A specification is, in fact, a strengthening of the notion of system of proper regular conditional probabilities. Indeed, in the former, the consistency condition (II) is required to hold for *every* configuration $\omega \in \Omega$, and not only for almost every $\omega \in \Omega$. This is because the notion of specification is defined without any reference to a particular measure.

A probability measure μ on $(\Omega, \mathcal{F})$ is said to be *consistent* with a specification γ if the latter is a realization of its finite-volume conditional probabilities, that is, if $\mu[A|\mathcal{F}_{\Lambda^c}](\,\cdot\,) = \gamma_\Lambda(A|\,\cdot\,)$ μ-a.s. for all $A \in \mathcal{F}$ and $\Lambda \in \mathcal{S}$. Equivalently, μ is consistent with γ if it satisfies the *DLR equation* (for Dobrushin, Lanford and Ruelle):

$$\mu \;=\; \mu\,\gamma_\Lambda \tag{2.10}$$

for each $\Lambda \in \mathcal{S}$. The right-hand side is the composed measure: $(\mu\,\gamma_\Lambda)(f) = \int \gamma_\Lambda(f|\omega)\,\mu(d\omega)$ for f bounded measurable. We denote $\mathcal{G}(\gamma)$ the set of measures consistent with γ (measures *specified* by γ). The description of this set is, precisely, the central issue in equilibrium statistical mechanics. A specification γ is *quasilocal* if for each $\Lambda \in \mathcal{S}$ and each f local, $\gamma_\Lambda f$ is a quasilocal function. Analogously, the specification is *quasilocal in the direction* θ if so are the functions $\gamma_\Lambda f$ for local f and finite Λ. A probability measure μ is *quasilocal* if it is consistent with some quasilocal specification. Gibbsian specifications — defined through interactions via Boltzmann's prescription— are the archetype of quasilocal specifications. Every Gibbs measure —i.e. every measure consistent with a Gibbsian specification— is quasilocal, and the converse requires only the additional property of non-nullness [22, 32]. Reciprocally, a sufficient condition for non-Gibbsianness is the existence of an *essential non-quasilocality* (essential discontinuity), that is, of a configuration at which *every* realization of some finite-volume conditional probability of μ is discontinuous. These discontinuities are related to the existence of phase transitions in some constrained systems [15, 17, 8].

For a specification γ let Ω_γ be the set of configurations where $\gamma_\Lambda f$ is continuous for all $\Lambda \in \mathcal{S}$ and all f local, and Ω_γ^θ the set of configurations for which all the functions $\gamma_\Lambda f$ are quasilocal in the direction θ.

Definition 2.11.

1. *A probability measure is* **almost quasilocal in the direction** θ *if it is consistent with a specification* γ *such that* $\mu(\Omega_\gamma^\theta) = 1$.

2. *A probability measure* μ *is* **almost quasilocal** *if it is consistent with a specification* γ *such that* $\mu(\Omega_\gamma) = 1$.

2.2 Thermodynamic functions and the variational principle

The variational principle links statistical mechanical and thermodynamical quantities. Rigorously speaking, the functions defined below (pressure and entropy density) do not quite correspond to standard thermodynamics. The corresponding notions in the latter depend only of a few parameters, while the objects below are functions on infinite dimensional spaces. These functions are, however, more informative from the probabilistic point of view, because, at least in the Gibbsian case, they are related to large-deviation principles. Translation invariance plays an essential role in the thermodynamic formalism. That is, we assume that there is an action ("translations") $\{\tau_i : i \in \mathbb{Z}^d\}$ on $\mathbb{L}$ which defines corresponding actions on configurations —$(\tau_i \omega)_x = \omega_{\tau_{-i}x}$—, on functions — $\tau_i f(\omega) = f(\tau_{-i}\omega)$—, on measures —$\tau_i \mu(f) = \mu(\tau_{-i} f)$— and on specifications —$(\tau_i \gamma)_\Lambda (f|\omega) = \gamma_{\tau_{-i}\Lambda}(\tau_{-i} f | \tau_{-i}\omega)$. [To simplify the notation we will write $\mu(f)$ instead of $E_\mu(f)$.] Translation invariance means invariance under all actions τ_i. We consider, in this section, only translation-invariant probability measures on Ω, whose space we denote by $\mathcal{M}^+_{1,\mathrm{inv}}(\Omega)$. We denote $\mathcal{G}_{\mathrm{inv}}(\gamma) = \mathcal{G}(\gamma) \cap \mathcal{M}^+_{1,inv}(\Omega)$. Furthermore, the convergence along subsets of $\mathbb{L}$ is restricted to sequences of cubes $\Lambda_n = \{\tau_{-i}(0) : i \in ([-n, n] \cap \mathbb{Z})^d\}$.

Definition 2.12 ([30]). *A measure $\nu \in \mathcal{M}^+_{1,inv}(\Omega)$ is* **asymptotically decoupled** *if there exist functions $g : \mathbb{N} \longrightarrow \mathbb{N}$ and $c : \mathbb{N} \longrightarrow [0, \infty)$ such that*

$$\lim_{n\to\infty} \frac{g(n)}{n} = 0 \qquad \textit{and} \qquad \lim_{n\to\infty} \frac{c(n)}{|\Lambda_n|} = 0 \, , \tag{2.13}$$

such that for all $n \in \mathbb{N}$, $A \in \mathcal{F}_{\Lambda_n}$ and $B \in \mathcal{F}_{(\Lambda_{n+g(n)})^c}$,

$$e^{-c(n)}\, \nu(A)\, \nu(B) \;\leq\; \nu(A \cap B) \;\leq\; e^{c(n)}\, \nu(A)\, \nu(B) \, . \tag{2.14}$$

This class of measures strictly contains the set of all Gibbs measures. In particular, as we observe below, it includes measures obtained by block transformations of Gibbs measures, many of which are known to be non-Gibbsian.

For $\mu, \nu \in (\Omega, \mathcal{F})$, the *relative entropy* at volume $\Lambda \in \mathcal{S}$ of μ relative to ν is defined as

$$H_\Lambda(\mu|\nu) \;=\; \begin{cases} \displaystyle\int_\Omega \frac{d\mu_\Lambda}{d\nu_\Lambda} \log \frac{d\mu_\Lambda}{d\nu_\Lambda}\, d\nu & \text{if } \mu_\Lambda \ll \nu_\Lambda \\ +\infty & \text{otherwise.} \end{cases} \tag{2.15}$$

The notation μ_Λ refers to the projection (restriction) of μ to $(\Omega_\Lambda, \mathcal{F}_\Lambda)$. The *relative entropy density* of μ relative to ν is the limit

$$h(\mu|\nu) \;=\; \lim_{n\to\infty} \frac{H_{\Lambda_n}(\mu|\nu)}{|\Lambda_n|} \tag{2.16}$$

provided it exists. The limit is known to exist if $\nu \in \mathcal{M}^+_{1,inv}(\Omega)$ is a Gibbs measure (and $\mu \in \mathcal{M}^+_{1,inv}(\Omega)$ arbitrary) and, more generally [30], if ν is asymptotically decoupled. In these cases $h(\,\cdot\,|\nu)$ is an affine non-negative function on $\mathcal{M}^+_{1,inv}(\Omega)$. For $\nu \in \mathcal{M}^+_{1,inv}(\Omega)$ and f a bounded measurable function, the *pressure* (or minus free-energy density) for f relative to ν is defined as the limit

$$p(f|\nu) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|^d} \log \int \exp\Big(\sum_{x\in\Lambda_n} \tau_x f\Big)\, d\nu$$

whenever it exists. This limit exists, for every quasilocal function f, if ν is Gibbsian or asymptotically decoupled [30], yielding a convex function $p(\,\cdot\,|\nu)$. For our purposes, it is important to distinguish between thermodynamical (specification-independent) and statistical mechanical (specification-dependent) variational principles.

Definition 2.17 (Specification-independent variational principle). *A measure $\nu \in \mathcal{M}^+_{1,inv}(\Omega)$ satisfies a variational principle if the relative entropy $h(\mu|\nu)$ and the pressure $p(f|\nu)$ exist for all $\mu \in \mathcal{M}^+_{1,inv}(\Omega)$ and all $f \in \mathcal{F}_{loc}$, and they are conjugate convex functions in the sense that*

$$p(f|\nu) \;=\; \sup_{\mu\in\mathcal{M}^+_{1,\mathrm{inv}}(\Omega)} \Big[\mu(f) - h(\mu|\nu)\Big] \tag{2.18}$$

for all $f \in \mathcal{F}_{loc}$, and

$$h(\mu|\nu) \;=\; \sup_{f\in\mathcal{F}_{loc}} \Big[\mu(f) - p(f|\nu)\Big] \tag{2.19}$$

for all $\mu \in \mathcal{M}^+_{1,inv}(\Omega)$.

Gibbs measures satisfy this specification-independent principle. Pfister [30, Section 3.1] has recently extended its validity to asymptotically decoupled measures. In these cases $h(\,\cdot\,|\nu)$ is the rate function for a (level 3) large-deviation principle for ν.

Definition 2.20 (Variational principle relative to a specification). *Let γ be a specification and $\nu \in \mathcal{G}_{\mathrm{inv}}(\gamma)$. We say that a variational principle occurs for ν and γ if for all $\mu \in \mathcal{M}^+_{1,inv}(\Omega)$*

$$h(\mu|\nu) = 0 \iff \mu \in \mathcal{G}_{\mathrm{inv}}(\gamma)\;. \tag{2.21}$$

The equivalence (2.21) holds for Gibbs measures ν, while the implication to the right is valid, more generally, for measures ν consistent with γ quasilocal (see [14], Chapter 10). In [25] the implication to the left was extended to block-transformed measures satisfying appropriate support hypothesis. Below we extend the implication to the right to some non-Gibbsian (non-quasilocal) measures.

2.3 Transformations of measures

Definition 2.22. *A* **renormalization transformation** T *from* $(\Omega, \mathcal{F})$ *to* $(\Omega', \mathcal{F}')$ *is a probability kernel* $T(\cdot|\cdot)$ *on* $(\Omega, \mathcal{F}')$. *That is, for each* $\omega \in \Omega$ $T(\cdot|\omega)$ *is a probability measure on* $(\Omega', \mathcal{F}')$ *and for each* $A' \in \mathcal{F}'$, $T(A'|\cdot)$ *is* $\mathcal{F}$-*measurable. The transformation is a* **block-spin transformation** *if* Ω' *is of the form* $(\Omega'_0)^{\mathbb{L}'}$ *and there exists* $\alpha > 0$ *(compression factor) such that the following two properties hold*

(i) Strict locality*: For every* n, $A' \in \mathcal{F}'_{\Lambda'_n}$ *implies* $T^{-1}(A') \in \mathcal{F}_{\Lambda'_{[\alpha n]}}$.

(ii) Factorization*: There exists a distance* $dist$ *in* $\mathbb{L}'$ *such that if* $A' \in \mathcal{F}'_{D'}$ *and* $B' \in \mathcal{F}'_{E'}$ *with* $dist(D', E') > \alpha$*, then* $T(A' \cap B'|\cdot) = T(A'|\cdot)\, T(B'|\cdot)$.

A renormalization transformation is *deterministic* if it is of the form $T(\cdot|\omega) = \delta_{t(\omega)}(\cdot)$ for some $t : \Omega \to \Omega'$. A renormalization transformation T induces a transformation $\mu \mapsto \mu T$ on measures, with $(\mu T)(f') = \mu[T(f')]$ for each $f' \in \mathcal{F}'$. In most applications, block-spin transformations have a product form: $T(d\omega'|\omega) = \prod_{x'} T_{x'}(d\omega_{x'}|\omega)$, where $T_x(\{\omega_{x'}\}|\cdot) \in \mathcal{F}_{B_{x'}}$, for a family of sets $\{B_{x'} \subset \mathbb{L} : x' \in \mathbb{L}'\}$ —the *blocks*— with bounded diameter whose union covers $\mathbb{L}$. Transformations of this sort are called *real-space renormalization transformations* in physics. The transformations defining cellular automata (with local rules) fit also into this framework. The corresponding blocks overlap and the compression factor may be chosen arbitrarily close to one. We briefly remind the reader of some of the transformations considered in the sequel:

- *Projections and decimations*: Given $D \subset \mathbb{L}$, this is the (product) deterministic transformation defined by $t(\omega) = (\omega_x)_{x \in D}$. The *decimation of spacing* $b \in \mathbb{N}$, for which $\mathbb{L} = \mathbb{Z}^d$, $D = b\mathbb{Z}^d$, is a block transformation, while *Schonmann's example* [31], corresponding to $D =$ hyperplane, is not because it fails to be strictly local.

- *Kadanoff*: This is a product block transformation defined by $T_x(d\omega_x|\omega) = \exp(p\,\omega'_x \sum_{y\in B_x} \omega_y)/\text{norm}$, for a given choice of parameter p and blocks B_x. In the limit $p \to \infty$ one obtains the *majority transformation* for the given blocks. If $B_x = bx$ this is a *noisy decimation*, that becomes the true decimation in the limit $p \to \infty$. More generally, one can define a *noisy projection* onto $D \subset \mathbb{L}$ through the transformation

$$\prod_{x\in D} \exp(p\,\omega'_x\omega_x)/\text{norm}.$$

It is well known that renormalization transformations can destroy Gibbsianness (for reviews see [5, 6, 9, 10]). Most of the non-Gibbsian measures resulting from block transformations were shown to be weakly Gibbsian [2, 29]. In Corollaries 3.5 and 3.6 below, we show that in some instances they are, in fact, almost quasilocal.

2.4 Monotonicity-preserving specifications

Finally we review notions related to stochastic monotonicity. Let us choose an appropriate (total) order "$\le$" for Ω_0 and, inspired by the case of the Ising model, let us call "plus" and "minus" the maximal and minimal elements. The choice induces a partial order on Ω: $\omega \le \sigma \iff \omega_x \le \sigma_x\ \forall x \in \mathbb{L}$. Its maximal and minimal elements are the configurations, denoted "$+$" and "$-$" in the sequel, respectively equal to "plus" and to "minus" at each site. For brevity, quasilocality in the "$+$", resp. "$-$", direction will be called *right continuity*, resp. *left continuity*. The partial order determines a notion of monotonicity for functions on Ω. A specification π is *monotonicity preserving* if for each finite $\Lambda \subset \mathbb{L}$, $\pi_\Lambda f$ is increasing whenever f is. These specifications have a number of useful properties. In the following lemma, we summarize those needed in the sequel. Proofs and more details can be found in [12].

Lemma 2.23. *Let γ be a monotonicity-preserving specification*

(a) *The limits $\gamma_\Lambda^{(\pm)}(\,\cdot\,|\omega) = \lim_{S\uparrow\mathbb{L}} \gamma_\Lambda(\,\cdot\,|\omega_S \pm_{S^c})$ exist and define two monotonicity-preserving specifications, $\gamma^{(+)}$ being right continuous and $\gamma^{(-)}$ left continuous. The specifications are translation-invariant if so is γ. Furthermore, $\gamma^{(-)}(f) \le \gamma(f) \le \gamma^{(+)}(f)$ for any local increasing f, and the specifications $\gamma^{(+)}$, $\gamma^{(-)}$ and γ are continuous on the set*

$$\Omega_\pm \;=\; \Big\{\omega \in \Omega : \gamma^{(+)}(f|\omega) = \gamma^{(-)}(f|\omega)\ \forall f \in \mathcal{F}_{loc}, \Lambda \in S\Big\}\,. \quad (2.24)$$

(b) *The limits* $\mu^{\pm} = \lim_{\Lambda\uparrow\mathbb{L}} \gamma_\Lambda(\,\cdot\,|\pm)$ *exist and define two extremal measures* $\mu^{\pm} \in \mathcal{G}(\gamma^{(\pm)})$ *[thus* μ^{+} *is right continuous and* μ^{-} *left continuous] which are translation-invariant if so is* γ*. If* f *is local and increasing,* $\mu^{-}(f) \le \mu(f) \le \mu^{+}(f)$ *for any* $\mu \in \mathcal{G}(\gamma)$*.*

(c) *For each (finite or infinite)* $D \subset \mathbb{L}$*, the conditional expectations* $\mu^{+}(f\,|\mathcal{F}_D)$ *and* $\mu^{-}(f\,|\mathcal{F}_D)$ *can be given everywhere-defined monotonicity-preserving right, resp left, continuous versions. In fact, these expectations come, respectively, from* global specifications*, that is, from families of stochastic kernels satisfying Definition 2.9 also for infinite* $\Lambda \subset \mathbb{L}$*. Furthermore,* $\mu^{-}(f\,|\mathcal{F}_D) \le \mu^{+}(f\,|\mathcal{F}_D)$ *for each* f *increasing.*

(d) *For each (infinite)* $D \subset \mathbb{L}$ *there exist monotonicity preserving specifications* $\Gamma^{(D,\pm)}$ *such that the projections* $\mu_D^{\pm} \in \mathcal{G}(\Gamma^{(D,\pm)})$ *and* $\Gamma_\Lambda^{(D,-)}(f) \le \Gamma_\Lambda^{(D,+)}(f)$ *for each* f *increasing. [By (a) and (c)* $\Gamma^{(D,+)}$ *(*$\Gamma^{(D,-)}$*) can be chosen to be right (left) continuous and extended to a global specification on* Ω_D *with the same properties.]*

Models satisfying the FKG property [13] are the standard source of monotonicity-preserving specifications. This class of models includes the ferromagnets with two- and one-body interactions (eg. Ising). Item (d) of the lemma is potentially relevant for renormalized measures because of the fact that a transformed measure μT can be seen as the projection on the primed variables of the measure $\mu \times T$ on $\Omega \times \Omega'$ defined by

$$(\mu \times T)(d\omega, d\omega') \;=\; T(d\omega'|\omega)\,\mu(d\omega)\;. \tag{2.25}$$

To apply (d) of the lemma, however, one has to find a suitable specification for this measure $\mu \times T$. If $\mu \in \mathcal{G}(\gamma)$ and T is a product transformation, a natural candidate is the family $\gamma \otimes T$ of stochastic kernels

$$(\gamma \otimes T)_{\Lambda\times\Lambda'}(d\omega_\Lambda, d\omega'_{\Lambda'}|\omega_{\Lambda^c}, \omega'_{(\Lambda')^c}) \;=$$

$$\mathrm{Norm}^{-1} \prod_{\substack{x'\notin\Lambda':\\ B_{x'}\cap\Lambda\neq\emptyset}} T_{x'}(\omega'_{x'}|\omega_{B_{x'}}) \prod_{x'\in\Lambda'} T_{x'}(d\omega'_{x'}|\omega_{B_{x'}})\,\gamma_\Lambda(d\omega_\Lambda|\omega_{\Lambda^c})\;.$$

Definition 2.26. *A pair* (γ, T)*, where* γ *is a specification and* T *a product renormalization transformation, is a* **monotonicity-preserving pair** *if the family* $\gamma \otimes T$ *is a monotonicity-preserving specification.*

It does not seem to be so simple to construct such monotonicity-preserving pairs. The only examples we know of are pairs for which $\gamma \otimes T$ is Gibbsian for a FKG interaction. This happens, for instance, for noisy projections (in particular noisy decimations) of the Ising measure.

3 Results

The following result follows immediately from Definitions 2.12 and 2.22.

Lemma 3.1. *If $\mu \in \mathcal{M}_1^+(\Omega)$ is asymptotically decoupled, then so is μT for every block-spin transformation T.*

From the results of Pfister, we can then conclude the following:

Theorem 3.2. *Let $\mu \in \mathcal{M}_{1,inv}^+(\Omega)$ be asymptotically decoupled and T be a block-spin transformation such that μT is translation-invariant. Then the renormalized measure μT satisfies the specification-independent variational principle of Definition 2.17.*

In [30, Section 3.4] it is showed that the relative entropy density $h(\,\cdot\,|\mu T)$ is the large deviation rate function of the empirical measure $L_\Lambda = \sum_{x\in\Lambda} \delta_{\tau_x\sigma}$. The next theorem states the criterion used in this paper to prove the implication to the right in (2.21) for non-quasilocal measures ν.

Theorem 3.3. *Let γ be a specification that is quasilocal in the direction $\theta \in \Omega$ and $\nu \in \mathcal{G}_{\rm inv}(\gamma)$. For each $\Lambda \in S$, $M \in \mathbb{N}$, $\Lambda \subset \Lambda_M$ and each local f, let $\gamma_\Lambda^{M,\theta}(f)$ denote the function $\omega \to \gamma_\Lambda(f|\omega_{\Lambda_M}\theta_{\mathbb{L}\setminus\Lambda_M})$. Then, if $\mu \in \mathcal{M}_{1,\rm inv}^+(\Omega)$ is such that $h(\mu|\nu) = 0$,*

$$\mu \in \mathcal{G}_{\rm inv}(\gamma) \iff \nu\Big[\frac{d\mu_{\Lambda_M\setminus\Lambda}}{d\nu_{\Lambda_M\setminus\Lambda}}\Big(\gamma_\Lambda^{M,\theta}(f) - \gamma_\Lambda(f)\Big)\Big] \underset{M\to\infty}{\longrightarrow} 0 \qquad (3.4)$$

for all $\Lambda \in S$ and $f \in \mathcal{F}_{loc}$.

We present two applications of the previous theorem. First we discuss systems with monotonicity-preserving specifications.

Corollary 3.5. *Consider a specification γ that is monotonicity preserving and translation invariant. Then, with the notation of Lemma 2.23,*

(a) *$h(\mu^-|\mu^+) = 0$ implies that $\mu^- \in \mathcal{G}(\gamma^{(+)})$ and $\mu^-(\Omega_{\gamma^{(+)}}) = \mu^-(\Omega_{\gamma^{(-)}}) = 1$ (hence μ^- is almost quasilocal).*

(b) *For $\mu \in \mathcal{M}_{1,\rm inv}^+(\Omega)$, $h(\mu|\nu^+) = 0$ and $\mu(\Omega_\pm) = 1$ imply that $\mu \in \mathcal{G}(\gamma^{(+)})$, and thus that μ almost quasilocal.*

Analogous results are valid interchanging "+" with "−".

By part (d) of Lemma 2.23, and the comments immediately thereafter, the preceding results apply when $\mu^{\pm}$ are the projections (possibly noisy) of the "plus" and "minus" phases of the Ising model. More generally, they can be the renormalized measures of the "plus" and "minus" measures of a monotonicity-preserving specification whenever the specification and the transformation form a monotonicity-preserving pair (Definition 2.26). At low temperature, the decimations (possibly noisy) μ^+T and μ^-T of the "plus" and "minus" phases of the Ising model are non-Gibbsian [17, 8], that is, all specifications with which these measures are consistent show essential discontinuities. The preceding corollary shows that, nevertheless, in these cases the implication to the right of the variational principle (2.21) can be recovered up to a point. If γ is a quasilocal translation-invariant specification and T a block-spin transformation, then $h(\mu T|\nu T) = 0$ for each $\mu, \nu \in \mathcal{G}_{\rm inv}(\gamma)$ such that μT and νT are translation invariant [8, formula (3.28)]. Hence, from part (a) of the previous corollary we conclude the following.

Corollary 3.6. *Let γ be a quasilocal, monotonicity-preserving, translation-invariant specification, and T a block-spin transformation that preserves translation invariance and such that the pair (γ, T) is monotonicity-preserving. Let $\mu^{\pm}$ be the extremal measures for γ [part (b) of Lemma 2.23] and $\pi^{(\pm)}$ be the right(left)-continuous specifications such that $\mu^{\pm}T \in \mathcal{G}_{\rm inv}(\pi^{\pm})$. Then*

$$\mu^- T \in \mathcal{G}(\pi^+) \quad \text{and} \quad \mu^- T(\Omega_{\pi^+}) = \mu^- T(\Omega_{\pi^-}) = 1 \tag{3.7}$$

(hence μ^-T is almost quasilocal). Analogous results are valid interchanging "+" with "−".

This corollary applies in particular for decimations (possibly noisy) of the Ising model. At low temperature, the renormalized measures μ^+T and μ^-T are in general non-Gibbsian [17, 8], that is, the specifications π^+ and π^- show essential discontinuities. Nevertheless, the preceding corollary, together with part (b) of Corollary 3.5 shows that in these cases the implication to the right of the variational principle (2.21) can be recovered, together with almost quasilocality. Several remarks are in order.

Remark 3.8. The preceding corollary strengthens, for (noisy) decimation transformations, the results of [2, 29] where only weak-Gibbsianness is proven. Our argument is apparently simpler than the renormalization and expansion-based procedures set up in these references, but, of course, it does not give such a complete description of the support of the decimated measures and it is only restricted to models with monotonicity properties.

Remark 3.9. For $d = 2$, the corollary implies that *all* the decimated measures of the Ising model are consistent with π^+ and almost quasilocal. This follows from the results of Aizenman [1] and Higuchi [16] showing that μ^+ and μ^- are the only extremal measures in $\mathcal{G}(\gamma)$.

Remark 3.10. Lefevere proves in [25] the implication to the left in (2.21), for ν a block-transformed measure and μ concentrated on an appropriate set $\widetilde{\Omega} \subset \Omega$ of ν-measure 1.

Our second application of Theorem 3.3 does not involve any monotonicity hypothesis. To formulate it we need some notation. For Λ a fixed finite volume, f a local function and $\epsilon > 0$, let

$$A_\epsilon^M(\theta, \Lambda, f) \;=\; \left\{\eta \in \Omega : \left|\gamma_\Lambda^{M,\theta}(f) - \gamma_\Lambda(f)\right| > \epsilon\right\} . \tag{3.11}$$

If γ is continuous in the direction θ, then $\mu(A_\epsilon^M) \longrightarrow 0$ as $M \uparrow \infty$ for any probability measure μ.

Proposition 3.12. *Let γ be a specification, $\nu \in \mathcal{G}(\gamma)$ and θ a configuration. Let us call a* ν-rate of θ-continuity *an increasing sequence of positive numbers $\alpha_M \uparrow \infty$ such that*

$$\limsup_{M\uparrow\infty} \frac{1}{\alpha_M} \log \nu[A_\epsilon^M(\theta, \Lambda, f)] \;<\; 0 \tag{3.13}$$

for all $\epsilon > 0$, f local and Λ finite. Then, a sufficient condition for a probability measure μ to be consistent with γ is the existence of a ν-rate of θ-continuity such that

$$\lim_{M\uparrow\infty} \frac{1}{\alpha_M} H_{\Lambda_M}(\mu|\nu) \;=\; 0 . \tag{3.14}$$

This proposition applies, for instance, to Schonmann's example. Indeed, if ν^+ is the projection on a (one-dimensional) layer of the low-temperature plus-phase of the two-dimensional Ising model, then the estimates in [27] imply that the monotone right-continuous specification γ^+ [such that $\nu^+ \in \mathcal{G}(\gamma^+)$] admits $\alpha_M = M$ as ν^+-rate of right-continuity. Hence for this example we can conclude that for any other measure μ on the layer, $h(\mu|\nu^+) = 0$ implies $\mu \in \mathcal{G}(\gamma^+)$. This is a strengthening of part (b) of Corollary 3.5. We emphasize that such a μ can *not* be the projection ν^- of the minus Ising phase. Indeed, while at present the existence of $h(\nu^-|\nu^+)$ has not rigorously been established, Schonmann's original argument [31] implies that $h(\nu^-|\nu^+) > 0$ if it exists.

4 Proofs

The proofs are basically transcriptions of those given in [11]. We include them for the sake of completeness.

4.1 Proof of Theorem 3.3

If $h(\mu|\nu)$ is defined then, for n sufficiently large there exists a $\mathcal{F}_{\Lambda_n}$-measurable function $g_{\Lambda_n} := d\mu_{\Lambda_n}/d\nu_{\Lambda_n}$. We fix a local f and $\Lambda \in \mathcal{S}$ and pick M such that $\Lambda_M \supset \Lambda$ and g_{Λ_M} exist. We have

$$\mu(\gamma_\Lambda f - f) \;=\; A_M + B_M + \text{RHS of (3.4)}\;, \tag{4.1}$$

with

$$A_M \;=\; \mu\Big[\gamma_{\Lambda_M}(f) - \gamma^{M,\theta}_{\Lambda_M}(f)\Big] \underset{M\to\infty}{\longrightarrow} 0 \tag{4.2}$$

by dominated convergence, because of the assumed quasilocality in the direction θ of γ, and

$$\begin{aligned} |B_M| \;&=\; \Big|\nu\Big[(g_{\Lambda_M} - g_{\Lambda_M\setminus\Lambda})\, f\Big]\Big| \\ &\le\; \sqrt{2\,\|f\|_\infty}\,\Big[H_\Delta(\mu|\nu) - H_{\Delta\setminus\Lambda}(\mu|\nu)\Big]^{1/2} \end{aligned}$$

for any $\Delta \supset \Delta_M$. The last bound is due to Csiszár's inequality [3]. This bound goes to zero as $\Delta \uparrow \mathbb{L}$, thanks to the hypothesis $h(\mu|\nu) = 0$, as shown in [14] or [30]. □

4.2 Proof of Corollary 3.5

It is enough to verify the right-hand side of (3.4) for increasing local functions f since linear combinations of these are uniformly dense in the set of quasilocal functions. Let us define

$$C_M(\mu) \;=\; \mu^+\Big[g_{\Lambda_M\setminus\Lambda}\Big(\gamma^{M,+}_\Lambda(f) - \gamma^{(+)}_\Lambda(f)\Big)\Big] \tag{4.3}$$

where $g_D = d\mu_D/d\mu_D^+$ for $D \subset \mathbb{L}$. In view of Theorem 3.3 consistency follows if $C_M \longrightarrow 0$ as $M \uparrow \infty$. In all cases, $0 \le C_M$ because γ is monotonicity preserving. We focus on upper bounds.

Part (a). It is a consequence of the following two relations. First,

$$\mu^+\Big(g_{\Lambda_M\setminus\Lambda}\, \gamma_\Lambda^{M,+}(f)\Big) \;=\; \mu^-\Big(\gamma_\Lambda^{M,+}(f)\Big) \tag{4.4}$$

because $\gamma_\Lambda^{M,+} f$ is $\mathcal{F}_{\Lambda_M\setminus\Lambda}$-measurable. Second

$$\begin{aligned}\mu^+\Big(g_{\Lambda_M\setminus\Lambda}\, \gamma_\Lambda^{(+)}(f)\Big) &= \mu^+\Big[g_{\Lambda_M\setminus\Lambda}\, \mu^+(\gamma_\Lambda(f)|\mathcal{F}_{\Lambda_M\setminus\Lambda})\Big]\\ &\ge \mu^+\Big[g_{\Lambda_M\setminus\Lambda}\, \mu^-(\gamma_\Lambda^{(+)}(f)|\mathcal{F}_{\Lambda_M\setminus\Lambda})\Big]\\ &= \mu^-\Big(\gamma_\Lambda^{(+)}(f)\Big)\,.\end{aligned} \tag{4.5}$$

The first and second lines are due to part (c) of Lemma 2.23 and the last one to the $\mathcal{F}_{\Lambda_M\setminus\Lambda}$-measurability of $\mu^-(\gamma_\Lambda^{(+)}(f)|\mathcal{F}_{\Lambda_M\setminus\Lambda})(\,\cdot\,)$.

We conclude that

$$C_M(\mu^-) \;\le\; \mu^-\Big(\gamma_\Lambda^{M,+}(f) - \gamma_\Lambda^{(+)}(f)\Big)$$

which converges to zero, as $M \to \infty$, because of the right-continuity of $\gamma^{(+)}$ and dominated convergence.

Part (b). We have

$$C_M(\mu) \;\le\; \mu^+\Big[g_{\Lambda_M\setminus\Lambda}\, \Big(\gamma_\Lambda^{M,+}(f) - \gamma_\Lambda^{M,-}(f)\Big)\Big] = \mu\Big(\gamma_\Lambda^{M,+}(f) - \gamma_\Lambda^{M,-}(f)\Big).$$

The inequality is by monotonicity and the equality by the $\mathcal{F}_{\Lambda_M\setminus\Lambda}$-measurability of $\gamma_\Lambda^{M,+}(f) - \gamma_\Lambda^{M,-}(f)$. Dominated convergence implies that the bound tends to zero with M, because $\mu(\Omega_\pm) = 1$. □

4.3 Proof of Proposition 3.12

Let us fix a local function f, a finite set Λ and some $\epsilon > 0$. We have

$$\nu\left[\frac{d\mu_{\Lambda_M\setminus\Lambda}}{d\nu_{\Lambda_M\setminus\Lambda}}\Big(\gamma_\Lambda^{M,\theta}(f) - \gamma_\Lambda(f)\Big)\right] \;\le\; \epsilon + 2\|f\|_\infty\, \widetilde{\mu}_M(A_\epsilon^M) \tag{4.6}$$

where

$$\widetilde{\mu}_M(A^M_\epsilon) \;=\; \nu\left(\frac{d\mu_{\Lambda_M\setminus\Lambda}}{d\nu_{\Lambda_M\setminus\Lambda}}\, 1_{A^M_\epsilon}\right) \,. \tag{4.7}$$

By (3.13) there exists $c > 0$ such that for M large enough,

$$\nu(A^M_\epsilon) \;\leq\; e^{-c\,\alpha_M} \;, \tag{4.8}$$

hence, for $0 < \;\delta < \; c$, and we can write the following inequalities:

$$\begin{aligned} \widetilde{\mu}_M(A^M_\epsilon) \;&\leq\; \frac{1}{\alpha_M\,\delta}\,\log\int \exp(\delta\,\alpha_M\,1_{A^M_\epsilon})\,d\nu \;+\; \frac{1}{\alpha_M\,\delta}\,H(\tilde{\mu}_M|\nu)\\ &\leq\; \frac{1}{\alpha_M\,\delta}\,\log\left(1+e^{\alpha_M\,\delta}\,\nu(A^M_\epsilon)\right) \;+\; \frac{1}{\alpha_M\,\delta}\,H(\tilde{\mu}_M|\nu)\\ &\leq\; \frac{1}{\alpha_M\,\delta}\,e^{\alpha_M(\delta-c)} \;+\; \frac{1}{\alpha_M\,\delta}\,H(\tilde{\mu}_M|\nu) \end{aligned} \tag{4.9}$$

where we have first used a standard upper bound of relative entropy obtained by Jensen's inequality ([18]). By (3.14), the last line tends to zero as $M \to \infty$. By (4.6), and the fact that $\epsilon > 0$ is arbitrary, we conclude that condition (3.4) of Theorem 3.3 is satisfied, which implies that $\mu \in \mathcal{G}(\gamma)$. □

Acknowledgements. We thank C.-E. Pfister for giving us an early version of [30], A. van Enter for comments and suggestions and E. Mahé for a correction. We are grateful to Eurandom (RF and ALN) and The University of Groningen (RF) for hospitality during the completion of this work.

References

[1] M. Aizenman. Translation invariance and instability of phase coexistence in the two dimensional Ising system. *Commun. Math. Phys.* **73:** (1980), 83–94.

[2] J. Bricmont, A. Kupiainen and R. Lefevere. Renormalization group pathologies and the definition of Gibbs states. *Commun. Math. Phys.* **194:** (1998), 359–388.

[3] I. Csiszár. Information-type measures of difference of probability distributions and indirect observations. *Studia Sci. Math. Hungar.* **2:** (1967), 299–318.

[4] R. L. Dobrushin and S.B. Shlosman. Gibbsian description of 'non Gibbsian' field. *Russian Math Surveys*, **52:** (1997), 285-297. Also 'Non Gibbsian' states and their description. *Commun. Math. Phys.* **200:** (1999), 125–179.

[5] A. C. D. van Enter. On the possible failure of the Gibbs property for measures on lattice systems. *Markov Proc. Rel. Fields* **2:** (1996), 209–224.

[6] A. C. D. van Enter. The renormalization-group peculiarities of Griffiths and Pearce: What have we learned?, in *Mathematical Results in Statistical Mechanics* (Proceedings of the colloquium with the same name, Marseille-Luminy, July 1998), S. Miracle-Solé, J. Ruiz and V. Zagrebnov, World Scientific, Singapore, 1999.

[7] A.C.D. van Enter, R. Fernández, F. den Hollander and F. Redig. Possible loss and recovery of Gibbsianness during the stochastic evolution of Gibbs measures. *Commun. Math. Phys.* **226:** (2002), 101–130.

[8] A.C.D. van Enter, R. Fernández and A.D. Sokal. Regularity properties and pathologies of position-space renormalization-group transformations: Scope and limitations of Gibbsian theory. *J. Statist. Phys.* **72:** (1993), 879–1167.

[9] R. Fernández. Random fields in lattices. The Gibbsianness issue. *Resenhas do IME-USP*, **3:** (1998), 391–421.

[10] R. Fernández. Measures for lattice systems. *Physica A* **263:** (1999), 117–130.

[11] R. Fernández, A. Le Ny and F. Redig. Variational principle and almost quasilocality for renormalized measures. *J. Statist. Phys.* **111:** (2003), 465–477.

[12] R. Fernández and C.-E. Pfister. Global specifications and non-quasilocality of projections of Gibbs measures. *Ann. Probab.* **25:** (1997), 1284–1315.

[13] C. M. Fortuin, J. Ginibre and P. W. Kasteleyn. Correlation inequalities on some partially ordered sets. *Commun. Math. Phys.* **22:** (1971), 89–103.

[14] H.O. Georgii. Gibbs Measures and Phase Transitions. Walter de Gruyter (de Gruyter Studies in Mathematics, Vol. 9), Berlin-New York, (1988).

[15] R.B. Griffiths and P.A. Pearce. Mathematicals properties of position-space renormalization-group transformations. *J. Statist. Phys.* **20:** (1979), 499–545.

[16] Y. Higuchi. On the absence of non-translationally invariant Gibbs states for the two-dimensional Ising system. In *Random Fields: Rigorous Results in Statistical Mechanics and Quantum Field Theory* (Esztergom 1979), J. Fritz, J. L. Lebowitz and D. Szasz (eds.), North Holland, Amsterdam, (1981).

[17] R.B. Israel. Banach algebras and Kadanoff transformations, in *Random Fields (Esztergom, 1979)* J. Fritz, J.L. Lebowitz and D.Szász, eds (North-Holland, Amsterdam, 1981), Vol II: (1981), 593–608.

[18] C. Kipnis and C. Landim. Scaling limits of interacting particle systems. Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), 320, Springer-Verlag, Berlin, (1999).

[19] C. Külske. (Non-)Gibbsianness and phase transitions in random lattice spin models. *Markov Proc. Rel. Fields* **5**(4): (1999), 357–383.

[20] C. Külske. Weakly Gibbsian representation for joint measures of quenched lattice spin models. *Probab. Theory Related Fields* **119:** (2001), 1–30.

[21] C. Külske, A. Le Ny and F. Redig. Relative entropy density and variational properties of generalized Gibbs measures, to appear in The Annals of Probability (2004).

[22] O. K. Kozlov. Gibbs description of a system of random variables. *Probl. Inform. Transmission.* **10:** (1974), 258–265.

[23] A. Le Ny. Mesures de Gibbs sur un réseau et non-gibbsiannité: restauration du formalisme gibbsien. Thèse de doctorat de l'université de Rennes 1, 2000. http://euridice.tue.nl/ãleny.

[24] A. Le Ny. Fractal Failure of Quasilocality for a Majority Rule Transformation on a Tree, *Lett. Math. Phys.* **54:** (2000), 11–24.

[25] R. Lefevere. Variational principle for some renormalized measures. *J. Statist. Phys.* **95:** (1999), 785–803.

[26] R. Lefevere. Almost and weak Gibbsianness: A long-range pair-interaction example. *J. Statist. Phys.* **96:** (1999), 109–113.

[27] C. Maes, F. Redig and A. Van Moffaert. Almost Gibbsian versus Weakly Gibbsian. *Stoch. Proc. Appl.* **79:** (1999), 1–15.

[28] C. Maes, F. Redig and A. Van Moffaert. The restriction of the Ising model to a Layer. *J. Statist. Phys.* **96:** (1999), 69–107.

[29] C. Maes, F. Redig, S. Shlosman and A. van Moffaert. Percolation, path large deviations and weak Gibbsianity. *Commun. Math. Phys.* **209:** (2000), 517–545.

[30] C.-E. Pfister. Thermodynamical aspects of classical lattice systems. Proceedings of the 4th brazilian school of probability, Mambucaba, RJ, 2000. In: *In and out of equilibrium. Probability with a physical flavour*. Progress in probability, Vladas Sidoravicius editor, Birkhaüser, pp. 393–472.

[31] R. H. Schonmann. Projections of Gibbs measures may be non-Gibbsian. *Commun. Math. Phys.* **124:** (1989), 1–7.

[32] W.G. Sullivan. Potentials for almost Markovian random fields. *Commun. Math. Phys.* **33:** (1973), 61–74.

Roberto Fernández
Laboratoire de Mathématiques Raphël Salem
Université de Rouen, UFR Sciences
site Colbert, 76821 Mont Saint-Aignan
FRANCE
E-mail: Roberto.Fernandez@univ-rouen.fr

Arnaud Le Ny
Université de Paris-sud
Laboratoire de Mathématiques
Batiment 425
91405 Orsay cedex
FRANCE
E-mail: arnaud.leny@math.u-psud.fr

Frank Redig
Faculteit Wiskunde en Informatica TU
Eindhoven, Postbus 513, 5600 MB Eindhoven
THE NETHERLANDS
E-mail: f.h.j.redig@TUE.nl